Pre-class Warm-up!!!

Can you remember what Newton's law of cooling says? Does it say:

a. $\frac{dT}{dt} = k(t - A)$

b. $\frac{dT}{dt} = k(T - At)$

c. $\frac{dT}{dt} = k(T - A)$

d. $\frac{dT}{dt} = -k(T - A)$

e. None of the above

Section 1.5: Linear first order differential equations

We learn:

- what does a linear equation look like?
- How to solve them
- How to do questions about tanks of brine.

We don't need:

- The theoretical statement of Theorem 1 on page 50 about the existence and uniqueness of solutions

A linear differential equation is a linear combination of derivatives of y by functions of x, like

$$P_n(x)\, y^{(n)} + P_{n-1}(x)\, y^{(n-1)} + \dots P_1(x)\, y' + P_0(x)\, y = Q(x)$$

A first order linear d.e. has the form:

$$P_1(x)\, y' + P_0(x)\, y = Q(x)$$

and we can write it:

$$y' + P(x)\, y = Q(x) \quad \left(Q = \frac{\text{old } Q}{P_1} \right)$$

Question: which of the following are linear d.e.'s?

a. y' = x-y Yes No

b. y y' + e^x = x^{15} Yes No

c. y' + ye^x = x^{15} Yes No

How to solve dy/dx + P(x) y = Q(x) ?

We multiply both sides by $e^{\int P(x)\,dx}$

to get

$$\frac{d}{dx}\left(e^{\int P} y\right) = e^{\int P} y' + e^{\int P} P y = e^{\int P} Q$$

Integrate both sides with respect to x

$e^{\int P}$ is the integrating factor

Question: Solve dy/dx = x-y

Solution

$y' + y = x$ so $P = 1$, $\int P = x$,

I.F. $= e^{x}$

$$e^{x} y' + e^{x} y = x e^{x}$$

$$\frac{d}{dx}\left(e^{x} y\right) = x e^{x}$$

$$e^{x} y = \int x e^{x}\,dx$$

$$= x e^{x} - \int e^{x}\,dx = x e^{x} - e^{x} + C$$

Divide by e^{x} :

$$y = x - 1 + C e^{-x}$$

y = x - 1 + Ce^{-x}

Page 56 question 24.

Solve $(x^2 + 4)\, y' + 3xy = x$, $y(0) = 1$.

Solution: $y' + \frac{3x}{x^2+4} y = \frac{x}{x^2+4}$

The Integrating Factor is: $e^{\int \frac{3x}{x^2+4} dx}$

$= e^{\frac{3}{2}\ln(x^2+4)} = e^{\ln(x^2+4)^{3/2}}$

$= (x^2+4)^{3/2}$

Multiply, to get

$(x^2+4)^{3/2} y' + 3x(x^2+4)^{\frac{1}{2}} y = x(x^2+4)^{\frac{1}{2}}$

$\frac{d}{dx}\left[(x^2+4)^{3/2} y\right] = x(x^2+4)^{1/2}$

$(x^2+4)^{3/2} y = \frac{1}{3}(x^2+4)^{3/2} + C$

$y = \frac{1}{3} + \frac{C}{(x^2+4)^{3/2}}$

$1 = \frac{1}{3} + \frac{C}{8} = y(0)$

$C = \frac{16}{3}$

We get $y = \frac{1}{3} + \frac{16}{3(x^2+4)^{3/2}}$

Question: Which method would you use to solve the differential equation

$$\frac{dy}{dx} = ye^{x}$$

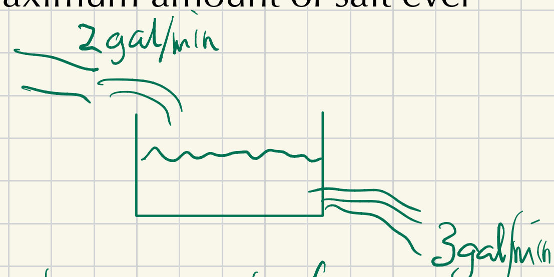
a. Separate the variables

b. Treat it as a linear first order equation

c. Do something else

Both work

Page 54 question 36.
A tank contains 60 gallons of pure water. Brine with concentration 1 lb salt per gallon enters at 2 gal/min. Perfectly mixed solution leaves at 3 gal/min. Thus the tank is empty after 1 hour.
(a) Find the amount of salt in the tank after t minutes,
(b) What is the maximum amount of salt ever in the tank?



Solution :

Let $x(t)$ be the amount of salt in the tank at time t.

The volume of liquid in the tank at time t is $60-t$

The concentration of salt at time t is $\dfrac{x(t)}{60-t}$ lb/gal

We get $\dfrac{dx}{dt} = 2 - \dfrac{3x}{60-t}$

$$\frac{dx}{dt} + \frac{3x}{60-t} = 2$$

$$\text{I.F.} = e^{\int \frac{3}{60-t}\,dt} = e^{-3\ln(60-t)}$$

$$= e^{\ln(60-t)^{-3}} = \frac{1}{(60-t)^3}$$

$$\frac{1}{(60-t)^3}\frac{dx}{dt} + \frac{3x}{(60-t)^4} = \frac{2}{(60-t)^3}$$

$$\frac{d}{dt}\frac{x}{(60-t)^3} = \frac{2}{(60-t)^3}$$

$$\frac{d}{dt}\frac{x}{(60-t)^3} = \frac{2}{(60-t)^3}$$

$$\frac{x}{(60-t)^3} = \frac{1}{(60-t)^2} + C$$

$$x = 60-t + C(60-t)^3$$

$$x(0) = 0 = 60 + C\,60^3$$

$$C = -\frac{1}{60^2}$$

$$x = 60-t - \frac{(60-t)^3}{60^2}$$

To find the maximum solve

$$\frac{dx}{dt} = 0$$